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On a Counterexample in Monotone Approximation

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Using some new ideas and careful calculation, the present paper shows that there exists a function $f \in C^k_{[-1,1]} \cap \Delta^k$ such that $\limsup_{n\to\infty} E_n^{(k)}(f)/\omega_{k+3}(f, n^{-1}) = +\infty$, which improves the result from Wu and Zhou (*in* "Progress in Approximation Theory" (P. Nevai and A. Pinkus, Eds.), pp. 857–866, Academic Press, New York, 1991). © 1992 Academic Press, Inc.

1. INTRODUCTION

Denote by $C^N_{[-1,1]}$ the class of functions which have N continuous derivatives on the interval $[-1, 1]$, and by Π_n the class of algebraic polynomials of degree at most n,

$$\Delta^k = \{f : \Delta_h^k f(x) \geq 0, x \in [-1, 1], x + kh \in [-1, 1]\},$$

where

$$\Delta_h^k f(x) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x + jh).$$

Let

$$\|f\| = \max_{-1 \leq x \leq 1} |f(x)| \qquad \text{for} \quad f \in C_{[-1,1]},$$

$$E_n^{(k)}(f) = \min\{\|f - p\| : p \in \Pi_n \cap \Delta^k\}$$

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for $f \in C_{[-1,1]} \cap \Delta^k$,

$$\omega_m(f, \delta) = \sup\{|\Delta_h^m f(x)| : x \in [-1, 1], x+h \in [-1, 1], 0 < h \leqslant \delta\}.$$

Recently, much research has been devoted to the study of monotone and comonotone approximation of functions by algebraic polynomials, especially to Jackson type estimates (cf., for example, [1–4, 6–8, 10–14, 17]). However, from the converse results of Lorentz and Zeller [9] and Shvedov [15], it appears that Jackson type estimates for higher degree moduli of smoothness do not hold true in monotone approximation. So in [16] we guessed that there exists a function $f \in C_{[-1,1]} \cap \Delta^k$ such that for $k \geqslant 1$,

$$\limsup_{n \to \infty} \frac{E_n^{(k)}(f)}{\omega_{k+2}(f, n^{-1})} = +\infty.$$

In [16], we showed a weaker result, which claims that there exists a function $f \in C^k_{[-1,1]} \cap \Delta^k$ such that

$$\limsup_{n \to \infty} \frac{E_n^{(k)}(f)}{\omega_{2k+1}(f, n^{-1})} = +\infty$$

for $k \geqslant 2$ or

$$\limsup_{n \to \infty} \frac{E_n^{(1)}(f)}{\omega_4(f, n^{-1})} = +\infty$$

for $k = 1$.

Using a new constructive method and careful calculation, the present paper improves the above result.

THEOREM. *Let $k \geqslant 1$. Then there exists a function $f \in C^k_{[-1,1]} \cap \Delta^k$ such that*

$$\limsup_{n \to \infty} \frac{E_n^{(k)}(f)}{\omega_{k+3}(f, n^{-1})} = +\infty.$$

2. PROOF OF THE THEOREM

LEMMA 1. *Suppose that $a > 0$, $\alpha(x) = a^2/(x^2 - a^2)$, $g_m(x, a) = x^m e^{\alpha(x)+1}$, and $x \in (-a, a)$, then*[1]

$$|g_m^{(m)}(x, a) - m!| \leqslant C(m) a^{-2} x^2, \qquad |x| < a.$$

[1]In the paper, $C(x)$ always indicates a positive constant depending upon x only, which may have different values in different places.

Proof. Lemma 1 is evidently true for $a/2 \leqslant |x| < a$. Now suppose $|x| < a/2$. Write

$$g_m^{(m)}(x, a) = m!\, e^{\alpha(x)+1} + e \sum_{j=1}^{m} (m-j)! \binom{m}{j} x^j \frac{d^j}{dx^j} e^{\alpha(x)}$$

$$= m! \left(1 + \frac{x^2}{x^2 - a^2} + O\left(\left(\frac{x^2}{x^2 - a^2}\right)^2\right)\right)$$

$$+ e \sum_{j=1}^{m} (m-j)! \binom{m}{j} x^j \frac{d^j}{dx^j} e^{\alpha(x)}. \tag{1}$$

It is easily verified that

$$\left|\frac{d}{dx} e^{\alpha(x)}\right| \leqslant C\, |x|\, a^{-2}. \tag{2}$$

By substituting $y = x/a$,

$$\left|\frac{d^j}{dx^j} e^{\alpha(x)}\right| = a^{-j} \left|\frac{d^j}{dy^j} \exp\left(\frac{1}{y^2 - 1}\right)\right| \leqslant C(m) a^{-j}, \qquad 2 \leqslant j \leqslant m. \tag{3}$$

Lemma 1 is therefore completed by combining (1)–(3) together. ∎

LEMMA 2. *Suppose that* $\varepsilon_n \geqslant 0$, $\varepsilon_n \to 0$, $n \to \infty$, $\sigma > 0$. *Define*

$$\bar{g}_k(x, \varepsilon_n, \sigma) = \varepsilon_n^{2+2\sigma} g_k(x, \varepsilon_n) + x^{k+2} - \varepsilon_n^{2+2\sigma} x^k, \qquad x \in (-\varepsilon_n, \varepsilon_n);$$

then for sufficiently large n *and* $x \in (-\varepsilon_n, \varepsilon_n)$,

$$\bar{g}_k^{(k)}(x, \varepsilon_n, \sigma) \geqslant 0.$$

Proof. By Lemma 1, through the calculation

$$\bar{g}_k^{(k)}(x, \varepsilon_n, \sigma) = \varepsilon_n^{2+2\sigma} g_k^{(k)}(x, \varepsilon_n) - k!\, \varepsilon_n^{2+2\sigma} + \frac{(k+2)!}{2} x^2$$

$$\geqslant \frac{(k+2)!}{2} x^2 - O(x^2 \varepsilon_n^{2\sigma}),$$

then for sufficiently large n and $|x| < \varepsilon_n$,

$$\bar{g}_k^{(k)}(x, \varepsilon_n, \sigma) \geqslant 0. \quad ∎$$

LEMMA 3. *Let*

$$F_l(x)=\sum_{j=1}^{l} n_j^{-1/4} f_{n_j}(x),$$
$$Q_l(x)=q_l(x)+n_l^{-1/4}(x^{k+2}-n_l^{-5/2}x^k),$$

where

$$f_n(x)=\begin{cases} x^{k+2}-n^{-5/2}x^k, & |x|\geqslant n^{-9/8}, \\ \bar{g}_k(x, n^{-9/8}, 1/9), & |x|<n^{-9/8}, \end{cases}$$

$q_l(x)$ is the algebraic polynomial of best approximation of degree n_l to $F_{l-1}(x)$, and $\{n_l\}$ is a subsequence of natural numbers chosen by induction: Set n_1 to be some natural number N,

$$n_l \geqslant \|F_{l-1}^{(2k+8)}\|. \tag{4}$$

Then the estimates

$$\|F_l - Q_l\| \sim n_l^{-1/4}\, \|f_{n_l} - x^{k+2} + n_l^{-5/2}x^k\| \sim n_l^{-9k/8-11/4}, \tag{5}$$
$$Q_l^{(k)}(0) \leqslant -C(k) n_l^{-11/4} \tag{6}$$

hold.

Proof. It is not difficult to see that

$$\|F_{l-1}-q_l\| = O(\|F_{l-1}^{(2k+8)}\|\, n_l^{-2k-8}), \tag{7}$$

by Lemma 1 and a theorem on simultaneous approximation to continuous functions and their derivatives from Leviatan [5],

$$|q_l^{(k)}(0)| = |F_{l-1}^{(k)}(0) - q_l^{(k)}(0)| = O(\|F_{l-1}^{(2k+8)}\|\, n_l^{-k-8}). \tag{8}$$

From the expression

$$F_l(x)-Q_l(x)=F_{l-1}(x)-q_l(x)+n_l^{-1/4}(f_{n_l}(x)-x^{k+2}+n_l^{-5/2}x^k),$$

noting that

$$\|f_{n_l}(x)-x^{k+2}+n_l^{-5/2}x^k\| = \max_{-n_l^{-9/8}<x<n_l^{-9/8}} \left| e n_l^{-5/2} x^k \exp\left(\frac{n_l^{-9/4}}{x^2-n_l^{-9/4}}\right)\right|$$
$$\sim n_l^{-9k/8-5/2}$$

and (4), (7), we get (5). To prove (6), we see

$$Q_l^{(k)}(0) = q_l^{(k)}(0) - k!\,n_l^{-11/4},$$

thus we only need apply (4) and (8). ∎

LEMMA 4. *Under the conditions of Lemma 3, for any $r(x) \in \Pi_{n_l} \cap \Delta^k$ and large enough l we have*

$$\|F_l - r\| \geqslant C(k) n_l^{-k-11/4}. \tag{9}$$

Proof. (5) and (6) imply that

$$n_l^{-9k/8} \|F_l - Q_l\| \leqslant C(k)\,|Q_l^{(k)}(0)| \leqslant C(k)\,|Q_l^{(k)}(0) - r^{(k)}(0)|, \tag{10}$$

by the Bernstein type inequality

$$\begin{aligned} |Q_l^{(k)}(0) - r^{(k)}(0)| &\leqslant C(k) n_l^k \,\|Q_l - r\| \\ &\leqslant C(k) n_l^k (\|Q_l - F_l\| + \|F_l - r\|). \end{aligned} \tag{11}$$

Combining (5), (10), and (11), for l large enough, we get (9). ∎

Proof of the Theorem. From Lemma 2, we see that there is an $N > 0$ such that for $n \geqslant N$,

$$f_n^{(k)}(x) \geqslant 0.$$

Now select $\{n_l\}$ by induction. Set $n_1 = N$,

$$n_{l+1} = 2(n_l^{4(k+3)} + [\|F_l^{(2k+8)}\|] + [\|F_l^{(k+3)}\|^5] + 1)$$

for $l = 1, 2, \ldots$, where $[x]$ is the greatest integer not exceeding x. Define

$$f(x) = \sum_{j=1}^{\infty} n_j^{-1/4} f_{n_j}(x).$$

It is clear that $f \in C^k_{[-1,1]} \cap \Delta^k$. For any $r \in \Pi_{n_l} \cap \Delta^k$,

$$\|f(x) - r(x)\| \geqslant \|F_l - r\| - \left\| \sum_{j=l+1}^{\infty} n_j^{-1/4} f_{n_j} \right\|.$$

Applying Lemma 4 we have

$$\|f(x) - r(x)\| \geqslant C(k)(n_l^{-k-11/4} - n_{l+1}^{-1/4}) \geqslant C(k)(n_l^{-k-11/4} - n_l^{-k-3}),$$

thus

$$E_{n_l}^{(k)}(f) \geqslant C(k) n_l^{-k-11/4}. \tag{12}$$

At the same time, in view of Lemma 3,

$$\begin{aligned}\omega_{k+3}(f, n_l^{-1}) &\leqslant \|F_{l-1}^{(k+3)}\|\, n_l^{-k-3} + \omega_{k+3}(f_{n_l}(x) - x^{k+2} + n_l^{-5/2}x^k, n_l^{-1}) \\ &\quad + O\left(\sum_{j=l+1}^{\infty} n_j^{-1/4}\right) \\ &= O(n_l^{-k-14/5}) + O(n_l^{-9k/8-11/4}) + O(n_l^{-k-3}). \qquad (13)\end{aligned}$$

Take $s = \frac{1}{20}$; then from (12) and (13) for sufficiently large l it follows that

$$\frac{E_{n_l}^{(k)}(f)}{\omega_{k+3}(f, n_l^{-1})} \geqslant C(k) n_l^s.$$

The proof of the theorem is completed. ∎

3. Remarks

Remark 1. As we indicated in the introduction, the theorem still leaves a gap open, that is, whether or not the same result for ω_{k+3} is valid for ω_{k+2}. However, by using $n^{-5/4}g_k(x, n^{-9/8}) + x^{k+1} - n^{-5/4}x^k$ to replace $\bar{g}_n(x, n^{-9/8}, \frac{1}{9})$ in Lemma 2 and in the sequel, with almost the same proof, we can establish an alternative in the comonotone case:

Let $k \geqslant 1$. Then there exists a function $f \in C^k_{[-1,1]}$, which satisfies that $f^{(k)}(x) \geqslant 0$ for $x \in [0, 1]$ and $f^{(k)}(x) \leqslant 0$ for $x \in [-1, 0]$, such that

$$\limsup_{n \to \infty} \frac{e_n^{(k)}(f)}{\omega_{k+2}(f, n^{-1})} = +\infty,$$

where $e_n^{(k)}(f)$ is the best approximation of degree n to f by polynomials which are comonotone with it, that is, polynomials p such that $p^{(k)}(x) f^{(k)}(x) \geqslant 0$ for all $x \in [-1, 1]$.

Remark 2. By using a Nikol'skii type inequality instead of a Bernstein type inequality, with carefully chosen ε_n and σ, in a similar way to the proof of the theorem, we can prove the corresponding results in L^p space for $1 \leqslant p < \infty$:

Let $k \geqslant 1$, $1 \leqslant p < \infty$. Then there exists a function $f \in C^k_{[-1,1]} \cap \Delta^k$ such that

$$\limsup_{n \to \infty} \frac{E_n^{(k)}(f)_p}{\omega_{k+3+[1/p]}(f, n^{-1})_p} = +\infty,$$

where $E_n^{(k)}(f)_p$, $\omega_m(f, t)_p$ are the corresponding best monotone approximation of degree n and modulus of smoothness in L^p space.

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